11.3 Infinite Series

An infinite series is an infinite addition of numbers

$$a_1 + a_2 + a_3 + \ldots$$

ex/

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$$

$$1 + 1 + 1 + 1 + \ldots$$

$$1 - 1 + 1 - 1 + \ldots$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

Some can be associated with a "sum"

Some cannot.

Examine partial sums of the above series.

S_n : the nth partial sum
what we get when we add
the first n terms

ex for $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$

$$S_1 = 1$$

$$S_2 = 1+\frac{1}{2} = 1.5$$

$$S_3 = 1+\frac{1}{2}+\frac{1}{4} = 1.75$$

$$S_4 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 1.875$$

$$\vdots$$

$$S_{20} \approx 1+\frac{1}{2}+\dots+\frac{1}{2^{19}} \approx 1.999998$$

So probably reasonable to say

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots = 2$$

call an infinite series with a sum <u>convergent</u> otherwise it is <u>divergent</u>

ex

$1+1+1+1+\ldots$

convergent or divergent?

$S_1 = 1$

$S_2 = 1+1 = 2$

$S_3 = 1+1+1 = 3$

$S_4 = 1+1+1+1 = 4$

clearly does not approach any limit

$\Rightarrow$ divergent.

ex

$1-1+1-1+1-1+\ldots$

$S_1 = 1$

$S_2 = 1-1 = 0$

$S_3 = 1-1+1 = 1$

$S_4 = \qquad 0$

$S_5 = \qquad 1$

just alternates between 1 & 0

$\Rightarrow$ divergent.

Can be hard to tell if a given series is divergent or convergent.

ex Harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Is divergent, but it grows very slowly.

$$(A) \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

$$\quad \geq \quad \geq \quad \geq \quad \geq \quad \geq \quad \geq \quad \geq \quad \geq$$

$$(B) \quad \frac{1}{2} + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\frac{1}{2}} + \cdots$$

B is smaller than A, but B grows without bound.

divergent.

Important type of Series where it is easy to tell if it diverges or converges:

Geometric Series

$$a + ar + ar^2 + ar^3 + ar^4 + \dots$$

is called a geometric series with ratio r

(each term is obtained by multiplying the previous term by r - "the ratio")

Converges if and only if

$|r| < 1$

- The sum will be $\boxed{\dfrac{a}{1-r}}$

If $|r| < 1$

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1-r}$$

ex

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

$$= 1 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots$$

$a = 1$

$r = \frac{1}{2}$ so $\frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

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ex

$$1 + 1 + 1 + \cdots$$

$a = 1$

$r = 1$ since $|r| \not< 1$

divergent

$$1 - 1 + 1 - 1 + \cdots$$

$a = 1$

$r = -1 \Rightarrow$ divergent.

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ex

$$1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$$

$$= 1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots$$

$a = 1$

$r = \frac{1}{5}$ $= \frac{1}{1-\frac{1}{5}} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$

~~ex~~

$$\frac{2}{3^2} + \frac{2}{3^4} + \frac{2}{3^6} + \ldots$$

$$a = \frac{2}{3^2}$$

$r =$ what we multiply by each time $= \dfrac{1}{3^2}$

can also divide a term by previous:

$$a + ar + ar^2 + \ldots$$

$$\frac{ar^2}{ar} = r \qquad \frac{\frac{2}{3^6}}{\frac{2}{3^4}} = \frac{1}{3^2}$$

converges

$$= \frac{\cancel{a}\,\frac{2}{3^2}}{1 - \frac{1}{3^2}} = \frac{\frac{2}{9}}{1 - \frac{1}{9}} = \frac{\frac{2}{9}}{\frac{8}{9}} = \frac{2}{9} \cdot \frac{9}{8} = \frac{1}{4}$$

ex

$$\frac{5}{2^2} - \frac{5^2}{2^5} + \frac{5^3}{2^8} - \frac{5^4}{2^{11}} + \frac{5^5}{2^{14}} - \cdots$$

$$a = \frac{5}{2^2}$$

$$r = \frac{-\frac{5^2}{2^5}}{\frac{5}{2^2}} = \frac{-5^{\cancel{2}}}{2^{\cancel{5}3}} \cdot \frac{\cancel{2^2}}{\cancel{5}} = \frac{-5}{2^3} = -\frac{5}{8}$$

$\frac{5}{8} < 1 \Rightarrow$ convergent

$$\text{Sum} = \frac{\frac{5}{4}}{1 - -\frac{5}{8}} = \frac{\frac{5}{4}}{\frac{13}{8}} = \frac{5}{\cancel{4}} \cdot \frac{\cancel{8}\,2}{13} = \frac{10}{13}$$

ex
Can think of repeating decimals as geometric series

$0.1212\overline{12}$

$= 0.12 + 0.0012 + 0.000012 + \cdots$

$$= \frac{12}{100} + \frac{12}{10000} + \frac{12}{1000000} + \cdots$$

$$= \frac{12}{100} + \frac{12}{100^2} + \frac{12}{100^3} + \cdots$$

$$a = \frac{12}{100}$$

$$r = \frac{1}{100}$$

$$\frac{\frac{12}{100}}{1 - \frac{1}{100}} = \frac{\frac{12}{100}}{\frac{99}{100}} = \frac{12}{\cancel{100}} \cdot \frac{\cancel{100}}{99}$$

$$= \frac{12}{99} = \frac{4}{33}$$

ex A patient receives 6 mg of a certain drug daily.

Each day the body eliminates 30% of the drug in its system

After extended treatment estimate the amount of drug in their system.

key: Assume infinite treatments to take infinite sum.

drug from a treatment in body after n days is ~~$6(.3)$~~ $6\cdot(.7)^n$

$$6 + 6(.7) + 6(.7)^2 + 6(.7)^3 + \dots$$

$a = 6$

$r = .3$

$$\frac{6}{1-.7} = \frac{6}{.3} = 6\cdot\frac{10}{3} = \frac{60}{3}$$

$= 20$ mg